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Rate form of the Eshelby and Hill tensors

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Abstract

Expressions are derived for the rates of change of the $\boldsymbol{S}$ and $\boldsymbol{P}$ tensors for transformed homogeneous inclusions in an anisotropic comparison medium undergoing prescribed changes of its elastic moduli. General results are obtained for ellipsoids and then reduced to yield explicit expressions in terms of the Stroh eigenvalues for cylindrical and disk-shaped inclusions in anisotropic solids and for spherical inclusions in isotropic solids. Applications are illustrated by solving the rate problem for an inhomogeneity in a large volume of a comparison medium, which is shown to be readily adaptable to standard averaging techniques for predictions of rates of change of overall moduli of composite materials experiencing evolution of phase moduli.

Keywords: Inclusions; Anisotropic; Composite materials

1. Introduction

Micromechanical modeling of local fields and overall properties of heterogeneous solids is often based on solutions of problems involving homogeneous inclusions of ellipsoidal shape residing in a large volume of a comparison medium with a certain elastic stiffness $\boldsymbol{C}$. As shown by Eshelby (1957) for isotropic solids, and by Kinoshita and Mura (1971) for anisotropic solids, application of a uniform transformation strain $\boldsymbol{\mu}$ within such inclusions generates there a uniform transformation strain field $\boldsymbol{\epsilon} = \boldsymbol{S}\boldsymbol{\mu}$, where $\boldsymbol{S}$ is the Eshelby tensor. The existence of a uniform field allows the connection $\boldsymbol{S} = \boldsymbol{P}\boldsymbol{C}$, where $\boldsymbol{P} = (\boldsymbol{C}^* + \boldsymbol{C})^{-1}$ and $\boldsymbol{C}^*$ is the stiffness of the cavity containing the inclusion in $\boldsymbol{C}$ (Hill, 1965). Evaluation of $\boldsymbol{S}$ relies on integration of the Green's function, which is available in explicit form only for certain material symmetries (Mura, 1982; Ting and Lee, 1997). Numerical procedures have been developed for evaluation of $\boldsymbol{S}$ in general anisotropic solids, however, extensive calculations are required for acceptable accuracy of the results (Ghahremani, 1977; Gavazzi and Lagoudas, 1990).

Although elastic moduli of the constituents remain constant in most applications of composite materials, there are notable exception caused, for example, by changes in temperature or moisture content, by

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non-linear deformation, or by a phase transformation. Under such circumstances, the variable moduli of the constituents r can be regarded as functions of an evolution parameter t and represented by stiffness tensors $\boldsymbol{C}_r(t)$ with known rates of change $\partial C^r_{ijkl}(t)/\partial t$. Established procedures can be then used to define a comparison medium $\boldsymbol{C}(t)$ and its rate of change $\partial \boldsymbol{C}(t)/\partial t$ in terms of the constituent properties. Micromechanical analysis of such problems has apparently not been formulated.

The present paper derives expressions for the rate forms $\partial \boldsymbol{P}(t)/\partial t$ and $\partial \boldsymbol{S}(t)/\partial t$ of the tensors, in terms of known coefficients $\partial \boldsymbol{C}(t)/\partial t$. Section 2 reviews established procedures for evaluation of the $\boldsymbol{P}$ tensor for ellipsoidal inclusions in an anisotropic media. Section 3 presents derivation of the $\partial \boldsymbol{P}(t)/\partial t$ tensors for ellipsoidal inclusions in anisotropic and transversely isotropic solids. Section 4 focuses on derivation of closed forms of $\partial \boldsymbol{P}(t)/\partial t$ tensors for cylindrical inclusions in anisotropic and orthotropic solids, for spherical inclusions in isotropic solids and for disk-shaped inclusions in anisotropic solids. Finally, the results are applied to the solution of the Eshelby problem in solids with evolving elastic moduli and local eigenstrains. This solution is then extended to estimates of evolving overall moduli of composite materials.

2. Ellipsoidal inclusions in an anisotropic solid

Consider an infinitely extended homogeneous anisotropic material with an elastic stiffness tensor C_{ijkl}. Suppose that in an ellipsoidal domain Ω within the surface,

$$x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 \leqslant 1 \tag{1}$$

there is a prescribed uniform distribution of eigenstrain μ^*_{ij} or eigenstress

$$\lambda^*_{ij} = -C_{ijkl}\mu^*_{kl} \tag{2}$$

The resulting stress $\boldsymbol{\sigma}^*$ and strain $\boldsymbol{\epsilon}^*$ inside Ω are uniform and defined in terms of the Eshelby tensor $\boldsymbol{S}$, or the Hill polarization tensor $\boldsymbol{P}$,

$$\epsilon^*_{ij} = S_{ijkl}\mu^*_{kl} \qquad \epsilon^*_{ij} = -P_{ijkl}\lambda^*_{kl} \tag{3}$$

with connections,

$$\sigma^*_{ij} = C_{ijkl}\epsilon^*_{kl} + \lambda^*_{ij} \qquad \epsilon^*_{ij} = C^{-1}_{ijkl}\sigma^*_{kl} + \mu^*_{ij} \tag{4}$$

The fields outside Ω, denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, are no longer uniform, however, their values at points adjoining the interface $\delta\Omega$ can be found (Hill, 1961, 1972, 1983) in the form suggested by Laws (1975, 1977) and Walpole (1977),

$$\epsilon_{ij} - \epsilon^*_{ij} = \mathscr{P}_{ijkl}\lambda^*_{kl} \tag{5}$$

$$\mathscr{P}_{ijkl}(\boldsymbol{n}) = \frac{1}{4}\left(\Gamma^{-1}_{jk}n_in_l + \Gamma^{-1}_{ik}n_jn_l + \Gamma^{-1}_{jl}n_in_k + \Gamma^{-1}_{il}n_jn_k\right) \tag{6}$$

where $\boldsymbol{n}$ is the unit normal vector to the interface $\partial\Omega$, and Γ_{ik} is the Christoffel matrix,

$$\Gamma_{ik}(\boldsymbol{n}) = C_{ijkl}n_jn_l \tag{7}$$

The jump in the interface strain components (5) can be utilized in evaluation of the strain field (3) inside Ω. In particular, Walpole (1977) has proved that $\int_{\partial\Omega} x_kn_k\epsilon_{ij}\,\mathrm{d}S = 0$ where the integral is over the interface $\partial\Omega$, and ϵ_{ij} is evaluated on the matrix side. Then after multiplying (5) by x_pn_p, integrating over the surface of the ellipsoid, and using definition (3), one obtains,

$$P_{ijkl} = \frac{\int_{\partial\Omega} x_pn_p\mathscr{P}_{ijkl}\,\mathrm{d}S}{\int_{\partial\Omega} x_pn_p\,\mathrm{d}S} = \frac{1}{4\pi a_1a_2a_3}\int_{\partial\Omega} x_pn_p\mathscr{P}_{ijkl}\,\mathrm{d}S \tag{8}$$

The surface integral $\int_{\partial\Omega} x_p n_p \,\mathrm{d}S$ was evaluated as three times the volume integral $\int_\Omega \mathrm{d}V$ by applying the divergence theorem. The $\int_\Omega \mathrm{d}V$ is the volume of the ellipsoid, which is $4/3\pi a_1 a_2 a_3$. To evaluate the remaining surface integral, we use a simple change of variables, that transforms the ellipsoid into a unit sphere, i.e., $x_1 = a_1\xi_1$, $x_2 = a_2\xi_2$, $x_3 = a_3\xi_3$. Then by applying to the integral (8) the divergence theorem, transforming the volume integral over the ellipsoid to the volume integral over the sphere, and applying the divergence theorem again to revert to the surface integral, we have,

$$\begin{aligned} P_{ijkl} &= \frac{1}{4\pi a_1 a_2 a_3} \int_V (\mathscr{P}_{ijkl} x_p)_{,x_p} \,\mathrm{d}V = \frac{1}{4\pi} \int_{V(\xi)} (\mathscr{P}_{ijkl}\xi_p)_{,\xi_p} \,\mathrm{d}V(\boldsymbol{\xi}) = \frac{1}{4\pi} \int_{S(\xi)} \xi_p \xi_p \mathscr{P}_{ijkl} \,\mathrm{d}S(\boldsymbol{\xi}) \\ &= \frac{1}{4\pi} \int_{S(\xi)} \mathscr{P}_{ijkl} \,\mathrm{d}S(\boldsymbol{\xi}) \end{aligned} \tag{9}$$

where $V(\boldsymbol{\xi})$ is the volume of the unit sphere, $S(\boldsymbol{\xi})$ is the surface of the sphere, and by construction $\xi_p\xi_p = 1$ on that surface. To evaluate $\mathscr{P}_{ijkl}(\boldsymbol{n})$, components of the unit normal vector $\boldsymbol{n}$ are expressed in $\boldsymbol{\xi}$ coordinates,

$$n_i = \frac{x_i/a_i^2}{(x_1^2/a_1^4 + x_2^2/a_2^4 + x_3^2/a_3^4)^{1/2}} = \frac{\xi_i/a_i}{(\xi_1^2/a_1^2 + \xi_2^2/a_2^2 + \xi_3^2/a_3^2)^{1/2}} \tag{10}$$

where there is no summation on index i. Next, we perform the following coordinate transformations, $\tilde{\omega}_1 = \xi_1/a_1$, $\tilde{\omega}_2 = \xi_2/a_2$, $\tilde{\omega}_3 = \xi_3/a_3$, and then $\omega_i = \tilde{\omega}_i/\tilde{\omega}$, where $\tilde{\omega} = (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2)^{1/2}$. It is clear that under those transformations, the unit sphere in $\boldsymbol{\xi}$-space is transformed into the unit sphere in $\boldsymbol{\omega}$-space. The surface element $\mathrm{d}S(\boldsymbol{\xi})$ is transformed to a new surface element $\mathrm{d}S(\boldsymbol{\omega}) = (a_1a_2a_3)^{-1}\tilde{\omega}^{-3}\mathrm{d}S(\boldsymbol{\xi})$ (Mura, 1982). Also, in $\boldsymbol{\omega}$ coordinates $n_i = \omega_i$. Then it follows that,

$$P_{ijkl} = \frac{a_1a_2a_3}{4\pi} \int_{S(\omega)} \tilde{\omega}^3 \mathscr{P}_{ijkl}(\boldsymbol{\omega}) \,\mathrm{d}S(\boldsymbol{\omega}) \tag{11}$$

It can be easily proved that $\xi_i = \omega_i a_i/(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{1/2}$ (no summation on i), and thus, $\tilde{\omega}_i = \omega_i/(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{1/2}$. Hence, $\tilde{\omega} = 1/(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{1/2}$. Finally, the $\boldsymbol{P}$ tensor is written in the form,

$$P_{ijkl} = \frac{a_1a_2a_3}{4\pi} \int_{S(\omega)} \frac{1}{(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{3/2}} \mathscr{P}_{ijkl}(\boldsymbol{\omega}) \,\mathrm{d}S(\boldsymbol{\omega}) \tag{12}$$

A somewhat different way of deriving the $\boldsymbol{P}$ tensor was adopted by Kinoshita and Mura (1971). They first obtained the displacement field inside and outside the inclusion (1) subjected to the uniform eigenstress (2),

$$u_j(\boldsymbol{x}) = -\frac{\lambda_{kl}^* x_i}{4\pi} \mathscr{M}_{kijl}(\boldsymbol{x}) \tag{13}$$

where

$$\mathscr{M}_{kijl}(\boldsymbol{x}) = \int_{S(\omega,x)} \frac{a_1a_2a_3}{(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{3/2}} \Gamma_{kj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \,\mathrm{d}S(\boldsymbol{\omega}) \tag{14}$$

and $S(\boldsymbol{\omega}, \boldsymbol{x})$ is the subset of the unit sphere $S(\boldsymbol{\omega})$ where condition

$$(\boldsymbol{x}\cdot\boldsymbol{\omega})^2 \leqslant a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2 \tag{15}$$

is satisfied. Condition (15) implies that the section area of the inclusion domain Ω cut by a plane perpendicular to $\boldsymbol{\omega}$ and containing the point $\boldsymbol{x}$ is not zero. When the point $\boldsymbol{x} \in \Omega$, then $S(\boldsymbol{\omega}, \boldsymbol{x})$ becomes the unit sphere, i.e., $S(\boldsymbol{\omega}, \boldsymbol{x}) = S(\boldsymbol{\omega})$, and the integral in (14) becomes independent of $\boldsymbol{x}$,

$$\mathscr{M}_{kijl} = \int_{S(\boldsymbol{\omega})} \frac{a_1 a_2 a_3}{(a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{3/2}} \Gamma_{kj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \,\mathrm{d}S(\boldsymbol{\omega}) \quad \boldsymbol{x} \in \Omega \tag{16}$$

The strain field inside Ω is obtained from (13) as,

$$\epsilon_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*) = -\frac{1}{8\pi}(\mathscr{M}_{kijl} + \mathscr{M}_{kjil})\lambda_{kl}^* \quad \boldsymbol{x} \in \Omega \tag{17}$$

Comparing definitions (3) with (17), one finds,

$$P_{ijkl} = \frac{1}{16\pi}(\mathscr{M}_{kijl} + \mathscr{M}_{kjil} + \mathscr{M}_{lijk} + \mathscr{M}_{ljik}) \qquad S_{ijkl} = P_{ijmn}C_{mnkl} \tag{18}$$

where the components P_{ijkl} have been made symmetric under $k \leftrightarrow l$ due to symmetry of the λ_{kl}^* tensor. The definitions (12), (16), and (18) are identical. They also indicate that $\boldsymbol{P}$ is symmetric and positive definite.

3. Rate forms of the $\boldsymbol{P}$ tensor

3.1. Ellipsoidal inclusions in an anisotropic medium

If the elastic constants of the solid containing the inclusion are prescribed as functions of an evolution parameter t, in the form $\partial C_{ijkl}(t)/\partial t$, then the $\boldsymbol{P}$ and $\boldsymbol{S}$ tensors also become functions of an evolution parameter t. The objective here is to find rate forms or derivatives $\dot{P}_{ijkl}$, $\dot{S}_{ijkl}$ of the $\boldsymbol{P}$ and $\boldsymbol{S}$ tensors in terms of the components $C_{ijkl}(t)$ and their derivatives $\dot{C}_{ijkl}(t)$. The derivative of the $\boldsymbol{P}$ tensor is obtained from (18) as,

$$\dot{P}_{ijkl} = \frac{1}{16\pi}(\dot{\mathscr{M}}_{kijl} + \dot{\mathscr{M}}_{kjil} + \dot{\mathscr{M}}_{lijk} + \dot{\mathscr{M}}_{ljik}) \tag{19}$$

We may also find the derivative of $\boldsymbol{P}$ in terms of the derivative of the interfacial tensor $\mathscr{P}$. From (12), it follows that,

$$\dot{P}_{ijkl} = \frac{1}{4\pi}\int_{S(\boldsymbol{\omega})} \frac{a_1 a_2 a_3}{\zeta^3} \dot{\mathscr{P}}_{ijkl}(\boldsymbol{\omega}) \,\mathrm{d}S(\boldsymbol{\omega}) \tag{20}$$

where $\zeta = (a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{1/2}$.

To find $\dot{\mathscr{P}}_{ijkl}$ we first note that for any tensor $\boldsymbol{\Gamma}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Gamma}^{-1} = -\boldsymbol{\Gamma}^{-1}\dot{\boldsymbol{\Gamma}}\boldsymbol{\Gamma}^{-1} \tag{21}$$

since $(\mathrm{d}/\mathrm{d}t)(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{-1}) = \boldsymbol{0}$. The derivative of $\mathscr{M}_{kijl}$ tensor then follows from (16) as,

$$\dot{\mathscr{M}}_{kijl} = -\int_{S(\boldsymbol{\omega})} \frac{a_1 a_2 a_3}{\zeta^3} \Gamma_{ks}^{-1}\dot{\Gamma}_{st}\Gamma_{tj}^{-1}\omega_i\omega_l \,\mathrm{d}S(\boldsymbol{\omega}) \tag{22}$$

To obtain the form of (16) and (22) more suitable for numerical evaluations the following change of variables is usually employed (Mura, 1982),

$$\zeta_1 = a_1\omega_1/\zeta \quad \zeta_2 = a_2\omega_2/\zeta \quad \zeta_3 = a_3\omega_3/\zeta$$
$$\zeta = (a_1^2\omega_1^2 + a_2^2\omega_2^2 + a_3^2\omega_3^2)^{1/2}$$

Then (16) is written as,

$$\mathscr{M}_{kijl} = \int_{S(\boldsymbol{\zeta})} \Gamma_{kj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \,\mathrm{d}S(\boldsymbol{\zeta}) \tag{23}$$

where $\mathrm{d}S(\boldsymbol{\zeta}) = a_1 a_2 a_3\,\mathrm{d}S(\boldsymbol{\omega})/\zeta^3$. Since $\Gamma^{-1}_{kj}(\boldsymbol{\omega})\omega_i\omega_l$ is a homogeneous polynomial of degree 0 the factor ζ can be dropped in $\omega_i = \zeta_i\zeta/a_i$ and (23) can be written as,

$$\mathscr{M}_{kijl} = \int_{S(\zeta)} \Pi_{kijl}\left(\frac{\zeta_1}{a_1}, \frac{\zeta_2}{a_2}, \frac{\zeta_3}{a_3}\right) \mathrm{d}S(\boldsymbol{\zeta}) \tag{24}$$

$$\Pi_{kijl}(\boldsymbol{\omega}) = \Gamma^{-1}_{kj}(\boldsymbol{\omega})\omega_i\omega_l \tag{25}$$

The surface element $\mathrm{d}S(\boldsymbol{\zeta})$ can be written as

$$\mathrm{d}S(\boldsymbol{\zeta}) = \mathrm{d}\zeta_3\,\mathrm{d}\psi \tag{26}$$

where

$$\zeta_1 = (1-\zeta_3^2)^{1/2}\cos\psi \quad \zeta_2 = (1-\zeta_3^2)^{1/2}\sin\psi \quad \zeta_3 = \zeta_3 \tag{27}$$

Then (24) takes the form,

$$\mathscr{M}_{kijl} = \int_{-1}^{1} \mathrm{d}\zeta_3 \int_0^{2\pi} \Pi_{kijl}\left(\frac{\zeta_1}{a_1}, \frac{\zeta_2}{a_2}, \frac{\zeta_3}{a_3}\right) \mathrm{d}\psi \tag{28}$$

We note that $\Gamma^{-1}_{ks}\dot{\Gamma}_{st}\Gamma^{-1}_{tj}\omega_i\omega_l$ is also a homogeneous polynomial of order 0 and thus one can write the derivative of the $\mathscr{M}_{kijl}$ tensor similarly to (28) as,

$$\dot{\mathscr{M}}_{kijl} = -\int_{-1}^{1} \mathrm{d}\zeta_3 \int_0^{2\pi} \Upsilon_{kijl}\left(\frac{\zeta_1}{a_1}, \frac{\zeta_2}{a_2}, \frac{\zeta_3}{a_3}\right) \mathrm{d}\psi \tag{29}$$

$$\Upsilon_{kijl}(\boldsymbol{\omega}) = \Gamma^{-1}_{ks}(\boldsymbol{\omega})\dot{\Gamma}_{st}(\boldsymbol{\omega})\Gamma^{-1}_{tj}(\boldsymbol{\omega})\omega_i\omega_l \tag{30}$$

Let $\hat{\mathbf{\Gamma}}$ signify the adjoint of $\mathbf{\Gamma}$, and $|\Gamma|$ the determinant of $\mathbf{\Gamma}$, i.e.,

$$\mathbf{\Gamma}\hat{\mathbf{\Gamma}} = |\Gamma|\boldsymbol{I} \tag{31}$$

Then (28) reads,

$$\mathscr{M}_{kijl} = \int_{-1}^{1} \mathrm{d}\zeta_3 \int_0^{2\pi} \frac{\widehat{\Gamma}_{kj}(\bar{\boldsymbol{\omega}})}{|\Gamma(\bar{\boldsymbol{\omega}})|} \bar{\omega}_i\bar{\omega}_l\,\mathrm{d}\psi \qquad \bar{\omega}_1 = \zeta_1/a_1 \qquad \bar{\omega}_2 = \zeta_2/a_2 \qquad \bar{\omega}_3 = \zeta_3/a_3 \tag{32}$$

and similarly (29) reads,

$$\dot{\mathscr{M}}_{kijl} = -\int_{-1}^{1} \mathrm{d}\zeta_3 \int_0^{2\pi} \frac{\widehat{\Gamma}^{-1}_{ks}(\bar{\boldsymbol{\omega}})\dot{\Gamma}_{st}(\bar{\boldsymbol{\omega}})\widehat{\Gamma}^{-1}_{tj}(\bar{\boldsymbol{\omega}})}{|\Gamma(\bar{\boldsymbol{\omega}})||\Gamma(\bar{\boldsymbol{\omega}})|} \bar{\omega}_i\bar{\omega}_l\,\mathrm{d}\psi \tag{33}$$

The forms (29) and (33) are suggested to be used for numerical evaluations of $\dot{\boldsymbol{P}}$ whenever the analytical expressions for the $\boldsymbol{P}$ tensor itself are unavailable or difficult to obtain. For a general anisotropic material Gavazzi and Lagoudas (1990) developed a procedure based on Gaussian quadrature formula to evaluate the double integral (32). The same procedure can be applied for evaluation of the integral (33).

As another alternative, Mura (1982) suggests to evaluate the integral with respect to the angle ψ by the use of theorem of residues after the following change of variables is performed,

$$\cos\psi = (z + z^{-1})/2 \quad \sin\psi = (z - z^{-1})/(2\mathrm{i}) \quad \mathrm{d}\psi = \mathrm{d}z/(\mathrm{i}z)$$

The residues will be in general dependent upon a value of ζ_3. However, more explicit results may be obtained for transversely isotropic materials with the axis of rotational symmetry aligned with the x_3-axis.

3.2. Spheroidal inclusion in a transversely isotropic medium

Let x_3 be the axis of elastic symmetry coincident with the axis of the inclusion (1). We use contracted notation C_{pq} for the components of the stiffness tensor C_{ijkl}, i.e., $C_{iiii} = C_{ii}$, $C_{iijj} = C_{ij}$, and $C_{2323} = C_{44}$, $C_{1313} = C_{55}$, $C_{1212} = C_{66}$. In contracted notation C_{pq}, the only non-zero components of the stiffness matrix C_{pq} are,

$$\begin{aligned} &C_{11} = C_{22} = k + m \quad C_{33} = n \quad C_{13} = C_{23} = l \\ &C_{12} = k - m \quad C_{44} = C_{55} = p \quad C_{66} = \tfrac{1}{2}(C_{11} - C_{12}) = m \end{aligned} \tag{34}$$

where k, m, n, l, and p are Hill's elastic moduli.

It can be shown that for a spheroidal inclusion with $a_1 = a_2 = 1$ and $1/a_3^2 = c$ in a transversely isotropic medium, $|\Gamma|$ is a function of only ζ_3 and is independent of ψ,

$$|\Gamma(\zeta_3)| = (C_{44}\zeta_3^2 c + C_{66}(1-\zeta_3^2))[c\zeta_3^2(1-\zeta_3^2)(C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2) + c^2\zeta_3^4 C_{44}C_{33} + C_{11}C_{44}(1-\zeta_3^2)^2] \tag{35}$$

Thus, (32) assumes the form,

$$\mathscr{M}_{kijl} = \int_{-1}^{1} \frac{1}{|\Gamma(\zeta_3)|}\,\mathrm{d}\zeta_3 \int_0^{2\pi} \widehat{\Gamma}_{kj}(\bar{\boldsymbol{\omega}})\bar{\omega}_i\bar{\omega}_l\,\mathrm{d}\psi \tag{36}$$

We list the values of a few of the internal integrals of (36), denoted as $\mathscr{I}$, below,

$$\begin{aligned} \mathscr{I}_{3333} &= 2\pi\zeta_3^2 c(C_{44}\zeta_3^2 c + C_{66}(1-\zeta_3^2))(C_{44}\zeta_3^2 c + C_{11}(1-\zeta_3^2)) \\ \mathscr{I}_{1111} = \mathscr{I}_{2222} &= \frac{\pi}{4}(1-\zeta_3^2)[(C_{11}C_{44} + 3C_{66}C_{44})(1-\zeta_3^2)^2 + 4C_{44}C_{33}c^2\zeta_3^4 \\ &\quad + (C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{44} - 3C_{44}^2 - 3C_{66}C_{33})c\zeta_3^2(\zeta_3^2 - 1)] \end{aligned} \tag{37}$$

Analytical evaluation of the integrals over ζ_3 of type (36) for transversely isotropic medium has been presented in the papers of Withers (1989) and Mikata (2001).

Analogously, one can write the integral (33) as,

$$\dot{\mathscr{M}}_{kijl} = -\int_{-1}^{1} \frac{1}{|\Gamma(\bar{\boldsymbol{\omega}})|^2}\,\mathrm{d}\zeta_3 \int_0^{2\pi} \widehat{\Gamma}_{ks}^{-1}(\bar{\boldsymbol{\omega}})\dot{\Gamma}_{st}(\bar{\boldsymbol{\omega}})\widehat{\Gamma}_{tj}^{-1}(\bar{\boldsymbol{\omega}})\bar{\omega}_i\bar{\omega}_l\,\mathrm{d}\psi \tag{38}$$

The integral can be evaluated numerically by employing Gaussian integration rule.

4. Rate forms of *P* for special shapes of ellipsoidal inclusions

In this section we present the rate forms of the tensor ***P*** for long cylinders, spheres, and thin disks.

4.1. Cylindrical inclusion in an anisotropic medium

A cylindrical inclusion domain Ω can be obtained from (1) by letting $a_3 \to \infty$ so that Ω becomes,

$$x_1^2/a_1^2 + x_2^2/a_2^2 \leqslant 1 \quad -\infty < x_3 < \infty \quad \rho = a_1/a_2 \tag{39}$$

Kinoshita and Mura (1971) proved that for interior points $\boldsymbol{x} \in \Omega$ and $a_3 \to \infty$, the tensor $\mathscr{M}_{kijl}(\boldsymbol{x})$ in (14) tends to the constant value,

$$\mathscr{M}_{kijl} = 2\int_{s(\boldsymbol{\omega})} \frac{a_1a_2}{a_1^2\omega_1^2 + a_2^2\omega_2^2}\Pi_{kijl}(\omega_1,\omega_2,0)\,\mathrm{d}s(\boldsymbol{\omega}) \quad \Pi_{kijl}(\boldsymbol{\omega}) = \Gamma_{kj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \tag{40}$$

where $s(\boldsymbol{\omega})$ is the unit circle $s(\boldsymbol{\omega}) = \{\boldsymbol{\omega}|\omega_1^2 + \omega_2^2 = 1\}$.

The derivative of the $\mathscr{M}$ tensor for a cylindrical inclusion is then obtained from (21) and (40) as,

$$\dot{\mathscr{M}}_{kijl} = -2\int_{s(\boldsymbol{\omega})} \frac{a_1 a_2}{a_1^2\omega_1^2 + a_2^2\omega_2^2} \Upsilon_{kijl}(\omega_1, \omega_2, 0)\,\mathrm{d}s(\boldsymbol{\omega})$$
$$\Upsilon_{kijl}(\omega_1, \omega_2, \omega_3) = \Gamma_{ks}^{-1}(\boldsymbol{\omega})\dot{\Gamma}_{st}(\boldsymbol{\omega})\Gamma_{tj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \tag{41}$$

To evaluate the integrals (40) and (41) we follow the procedure described by Ting and Lee (1997). Let $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{m}}$ be two fixed orthogonal unit vectors on the plane $\omega_3 = 0$; any unit vector $\boldsymbol{\omega}$ lying on that plane can be represented by,

$$\boldsymbol{\omega} = \hat{\boldsymbol{n}}\cos\psi + \hat{\boldsymbol{m}}\sin\psi \tag{42}$$

where $0 \leqslant \psi \leqslant 2\pi$. Then,

$$\Gamma_{kj}(\omega_1, \omega_2, 0) = C_{ksjt}\omega_s\omega_t|_{\omega_3=0} = C_{ksjt}(\hat{n}_s\cos\psi + \hat{m}_s\sin\psi)(\hat{n}_t\cos\psi + \hat{m}_t\sin\psi) \tag{43}$$

By introducing new tensors,

$$Q_{kj} = C_{ksjt}\hat{n}_s\hat{n}_t \qquad R_{kj} = C_{ksjt}\hat{n}_s\hat{m}_t \qquad T_{kj} = C_{ksjt}\hat{m}_s\hat{m}_t \tag{44}$$

the tensor $\Gamma_{kj}(\omega_1, \omega_2, 0)$ can be written as,

$$\boldsymbol{\Gamma}(\psi) = \boldsymbol{Q}\cos^2\psi + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}})\cos\psi\sin\psi + \boldsymbol{T}\sin^2\psi = \cos^2\psi\boldsymbol{\Gamma}(z) \tag{45}$$

where $z = \tan\psi$ and,

$$\boldsymbol{\Gamma}(z) = \boldsymbol{Q} + z(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + z^2\boldsymbol{T} \tag{46}$$

Let us now denote

$$\Lambda_{il}(\boldsymbol{\omega}) = \omega_i\omega_l \tag{47}$$

and express the tensor $\boldsymbol{\Lambda}(\omega_1, \omega_2, 0)$ in terms of a parameter ψ, in a manner similar to (45),

$$\boldsymbol{\Lambda}(\psi) = \boldsymbol{F}\cos^2\psi + (\boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}})\cos\psi\sin\psi + \boldsymbol{H}\sin^2\psi = \cos^2\psi\boldsymbol{\Lambda}(z) \tag{48}$$

where

$$F_{il} = \hat{n}_i\hat{n}_l \quad G_{il} = \hat{n}_i\hat{m}_l \quad H_{il} = \hat{m}_i\hat{m}_l \tag{49}$$

and

$$\boldsymbol{\Lambda}(z) = \boldsymbol{F} + z(\boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}}) + z^2\boldsymbol{H} \tag{50}$$

Also,

$$a_1^2\omega_1^2 + a_2^2\omega_2^2 = \cos^2\psi(a + 2bz + cz^2) \quad a = a_1^2\hat{n}_1^2 + a_2^2\hat{n}_2^2 \;\; b = (a_1^2\hat{n}_1\hat{m}_1 + a_2^2\hat{n}_2\hat{m}_2) \;\; c = a_1^2\hat{m}_1^2 + a_2^2\hat{m}_2^2 \tag{51}$$

Using the definitions (45), (48), and (51) the integral in (41) can be written as,

$$\dot{\mathscr{M}}_{kijl} = -4\int_{-\pi/2}^{\pi/2} \frac{a_1 a_2}{\cos^2\psi(a + 2bz + cz^2)} \Gamma_{ks}^{-1}(\psi)\dot{\Gamma}_{st}(\psi)\Gamma_{tj}^{-1}(\psi)\Lambda_{il}(\psi)\,\mathrm{d}\psi \tag{52}$$

or, in terms of the variable $z = \tan\psi$ as,

$$\dot{\mathscr{M}}_{kijl} = -4\int_{-\infty}^{\infty} \frac{a_1 a_2}{(a + 2bz + cz^2)} \Gamma_{ks}^{-1}(z)\dot{\Gamma}_{st}(z)\Gamma_{tj}^{-1}(z)\Lambda_{il}(z)\,\mathrm{d}z \tag{53}$$

since $\mathrm{d}\psi = \mathrm{d}z/(1 + z^2) = \cos^2\psi\,\mathrm{d}z$. Moreover, if $|\Gamma(z)|$ denotes the determinant of $\boldsymbol{\Gamma}(z)$ and $\hat{\boldsymbol{\Gamma}}(z)$ the adjoint of $\boldsymbol{\Gamma}(z)$,

$$\mathbf{\Gamma}(z)\hat{\mathbf{\Gamma}}(z) = |\Gamma(z)|\boldsymbol{I} \tag{54}$$

then (53) assumes the form,

$$\dot{\mathscr{M}}_{kijl} = -4\int_{-\infty}^{\infty} \frac{a_1 a_2 \widehat{\Gamma}_{ks}(z)\dot{\Gamma}_{st}(z)\widehat{\Gamma}_{tj}(z)}{(a+2bz+cz^2)|\Gamma(z)|^2}\Lambda_{il}(z)\,\mathrm{d}z \tag{55}$$

Suppose the unit vectors in (42) are $\hat{\boldsymbol{n}} = \{1 \quad 0 \quad 0\}^{\mathrm{T}}$, $\hat{\boldsymbol{m}} = \{0 \quad 1 \quad 0\}^{\mathrm{T}}$. Then $a = a_1^2$, $b = 0$, and $c = a_2^2$. Then the denominator of (55) is equal to zero if $z = \pm(a_1/a_2)\mathrm{i}$ or if the determinant of $\mathbf{\Gamma}(z)$ vanishes, i.e.,

$$|\Gamma(z)| = 0 \tag{56}$$

Eq. (56) is of the sixth order in z. Let z_v $(v = 1, 2, 3)$ denote the roots of (56) with a positive imaginary part. Now, in view of (46), the determinant of $\mathbf{\Gamma}(z)$ may be expressed by,

$$|\Gamma(z)| = |T|f(z) \tag{57}$$

where

$$f(z) = (z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)(z-z_3)(z-\bar{z}_3), \tag{58}$$

the overbar signifies complex conjugate and $|T|$ is the determinant of matrix $\boldsymbol{T}$. From the theorem of residues we then obtain,

$$\dot{\mathscr{M}}_{kijl} = -\frac{8\pi\mathrm{i}}{|T|^2}\left\{\frac{\widehat{\Gamma}_{ks}(\rho\mathrm{i})\dot{\Gamma}_{st}(\rho\mathrm{i})\widehat{\Gamma}_{tj}(\rho\mathrm{i})}{2\mathrm{i}f^2(\rho\mathrm{i})}\Lambda_{il}(\rho\mathrm{i}) + \sum_{v=1}^{3}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\widehat{\Gamma}_{ks}(z)\dot{\Gamma}_{st}(z)\widehat{\Gamma}_{tj}(z)\Lambda_{il}(z)}{(\rho+z^2/\rho)f^2(z)}(z-z_v)^2\right]\Bigg|_{z=z_v}\right\}$$
$$\rho = a_1/a_2 \tag{59}$$

It should be pointed out that (59) remains valid when z_v are all different and $z_v \neq \rho\mathrm{i}$. For other cases, an alternative expression may be obtained depending on the number and multiplicity of the poles of the integrand (55).

It is instructive here to obtain expressions for the components of the $\boldsymbol{P}$ tensor itself. From (40) we have,

$$\mathscr{M}_{kijl} = 4\int_{-\infty}^{\infty}\frac{a_1 a_2 \widehat{\Gamma}_{kj}(z)}{(a+2bz+cz^2)|\Gamma(z)|}\Lambda_{il}(z)\,\mathrm{d}z \tag{60}$$

and, again, by the use of the theorem of residues, we obtain,

$$\mathscr{M}_{kijl} = \frac{8\pi\mathrm{i}}{|T|}\left\{\frac{\widehat{\Gamma}_{kj}(\rho\mathrm{i})\Lambda_{il}(\rho\mathrm{i})}{2\mathrm{i}f(\rho\mathrm{i})} + \sum_{v=1}^{3}\frac{\widehat{\Gamma}_{kj}(z_v)\Lambda_{il}(z_v)}{(\rho+z_v^2/\rho)f'(z_v)}\right\} \quad \rho = a_1/a_2 \tag{61}$$

4.2. Cylindrical inclusion in an orthotropic material

For an orthotropic material, and the unit vectors in (42) equal to $\hat{\boldsymbol{n}} = \{1 \quad 0 \quad 0\}^{\mathrm{T}}$, $\hat{\boldsymbol{m}} = \{0 \quad 1 \quad 0\}^{\mathrm{T}}$, the matrix $\mathbf{\Gamma}(z)$ becomes,

$$\mathbf{\Gamma}(z) = \begin{pmatrix} C_{11}+z^2C_{66} & z(C_{12}+C_{66}) & 0 \\ z(C_{12}+C_{66}) & C_{66}+z^2C_{22} & 0 \\ 0 & 0 & C_{55}+z^2C_{44} \end{pmatrix} \tag{62}$$

and the matrix $\mathbf{\Lambda}(z)$ is,

$$\mathbf{\Lambda}(z) = \begin{pmatrix} 1 & z & 0 \\ z & z^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{63}$$

Contracted notation C_{pq}, $(p, q = 1, 2, \ldots, 6)$ for the elastic constants C_{ijkl} has been used in (62). One of the roots z_v of (56) is immediately found as $z_3 = \mathrm{i}\sqrt{C_{55}/C_{44}}$. Other two z_v with a positive imaginary part are chosen among four roots of (56) given by $\pm\lambda e^{\pm i\alpha}$ (Yang and Chou, 1976), where

$$
\begin{aligned}
& z_{1,2} = \pm\lambda e^{\pm i\alpha} \quad \mathrm{Im}(z_{1,2}) > 0 \\
& \lambda = (C_{11}/C_{22})^{1/4} \quad \alpha = \arccos(\sqrt{-C}/2) \quad \text{for} -4 < C \leqslant 0 \\
& \alpha = \arccos(\mathrm{i}\sqrt{C}/2) \quad \text{for } C > 0 \\
& C = \frac{C_{11}C_{22} - C_{12}^2 - 2C_{12}C_{66}}{C_{66}\sqrt{C_{11}C_{22}}} - 2 \quad -4 < C < \infty
\end{aligned} \tag{64}
$$

The only non-zero components of the adjoint of $\mathbf{\Gamma}(z)$ are found as,

$$
\begin{aligned}
& \widehat{\Gamma}_{11} = (C_{66} + z^2 C_{22})(C_{55} + z^2 C_{44}) \\
& \widehat{\Gamma}_{22} = (C_{11} + z^2 C_{66})(C_{55} + z^2 C_{44}) \\
& \widehat{\Gamma}_{12} = \widehat{\Gamma}_{21} = -z(C_{12} + C_{66})(C_{55} + z^2 C_{44}) \\
& \widehat{\Gamma}_{33} = (C_{11} + z^2 C_{66})(C_{66} + z^2 C_{22}) - z^2 (C_{12} + C_{66})^2
\end{aligned} \tag{65}
$$

The tensor $\dot{\boldsymbol{P}}$ is now evaluated by using (62)–(65) in (59) and (19); the tensor $\boldsymbol{P}$ itself is evaluated from (61) and (18). Below we give expressions for the components of the $\boldsymbol{P}$ tensor, as it follows from (61), for an elliptic cylinder in an orthotropic solid,

$$
\begin{aligned}
P_{11} = P_{1111} &= \frac{2\mathrm{i}}{|T|}\left\{ \frac{(C_{66} - \rho^2 C_{22})(C_{55} - \rho^2 C_{44})}{2\mathrm{i} f(\rho \mathrm{i})} + \sum_{v=1}^{3} \frac{(C_{66} + z_v^2 C_{22})(C_{55} + z_v^2 C_{44})}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \\
P_{12} = P_{1122} &= \frac{2\mathrm{i}(C_{12} + C_{66})}{|T|}\left\{ \frac{\rho^2 (C_{55} - \rho^2 C_{44})}{2\mathrm{i} f(\rho \mathrm{i})} - \sum_{v=1}^{3} \frac{z_v^2 (C_{55} + z_v^2 C_{44})}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \\
P_{22} = P_{2222} &= \frac{2\mathrm{i}}{|T|}\left\{ \frac{-\rho^2 (C_{11} - \rho^2 C_{66})(C_{55} - \rho^2 C_{44})}{2\mathrm{i} f(\rho \mathrm{i})} + \sum_{v=1}^{3} \frac{z_v^2 (C_{11} + z_v^2 C_{66})(C_{55} + z_v^2 C_{44})}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \\
P_{66} = 4P_{1212} &= \frac{2\mathrm{i}}{|T|}\left\{ \frac{(C_{11} + \rho^4 C_{22} + 2\rho^2 C_{12})(C_{55} - \rho^2 C_{44})}{2\mathrm{i} f(\rho \mathrm{i})} + \sum_{v=1}^{3} \frac{(C_{11} + z_v^4 C_{22} - 2z_v^2 C_{12})(C_{55} + z_v^2 C_{44})}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \\
P_{55} = 4P_{1313} &= \frac{2\mathrm{i}}{|T|}\left\{ \frac{(C_{11} - \rho^2 C_{66})(C_{66} - \rho^2 C_{22}) + \rho^2 (C_{12} + C_{66})^2}{2\mathrm{i} f(\rho \mathrm{i})} \right. \\
&\quad \left. + \sum_{v=1}^{3} \frac{(C_{11} + z_v^2 C_{66})(C_{66} + z_v^2 C_{22}) - z_v^2 (C_{12} + C_{66})^2}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \\
P_{44} = 4P_{2323} &= \frac{2\mathrm{i}}{|T|}\left\{ \frac{-\rho^2 [(C_{11} - \rho^2 C_{66})(C_{66} - \rho^2 C_{22}) + \rho^2 (C_{12} + C_{66})^2]}{2\mathrm{i} f(\rho \mathrm{i})} \right. \\
&\quad \left. + \sum_{v=1}^{3} \frac{z_v^2 [(C_{11} + z_v^2 C_{66})(C_{66} + z_v^2 C_{22}) - z_v^2 (C_{12} + C_{66})^2]}{(\rho + z_v^2/\rho) t_v(z_v)} \right\} \quad z_v \neq \mathrm{i}, \rho \mathrm{i}
\end{aligned} \tag{66}
$$

where

$$
t_1(z_1) = f'(z_1) = (z_1 - \bar{z}_1)(z_1 - z_2)(z_1 - \bar{z}_2)(z_1 - z_3)(z_1 - \bar{z}_3) \tag{67}
$$

and all other $t_v(z_v)$ are obtained from (67) by a cyclic permutation of the subscripts.

It is possible to simplify further (66) and give an expression that is valid also for degenerate case of transversely isotropic materials when all $z_v = \mathrm{i}$. To accomplish this we follow the procedure described in detail by Ting and Lee (1997).

It can be shown that for an orthotropic material the determinant (57) is a cubic function in z^2 and,

$$f(z) = (z^2 - z_1^2)(z^2 - z_2^2)(z^2 - z_3^2) \tag{68}$$

We also note that it is convenient to consider $\rho\mathrm{i}$ formally as a fourth root (with positive imaginary part) of the polynomial $f(z)(\rho^2 + z^2)/\rho$. Thus, summation in (66) over index v can be carried out from $v = 1$ to 4.

Roots of the Eq. (68) permit to use the notation,

$$z_1 + z_2 = \mathrm{i}g \quad z_1 z_2 = -h \quad z_1^2 + z_2^2 = -s \quad z_3 = \mathrm{i}\beta_3 \quad z_4 = \mathrm{i}\rho \tag{69}$$

where

$$\begin{aligned} g &= \sqrt{\frac{C_{11}C_{22} - C_{12}^2 + 2C_{66}((C_{11}C_{22})^{1/2} - C_{12})}{C_{22}C_{66}}} > 0 \\ \beta_3 &= \sqrt{C_{55}/C_{44}} \quad h = \sqrt{\frac{C_{11}}{C_{22}}} \quad s = g^2 - 2h \quad \rho = a_1/a_2 \end{aligned} \tag{70}$$

As it has been shown by Ting and Lee (1997), to simplify (66) one needs to collect the terms before z_v^n, $n = 0, 2, 4, 6$ and then for each exponent n sum these terms from $v = 1$ to 4. To represent the resulting quantities we introduce the following parameters,

$$\begin{aligned} n_0 &= 2(h\beta_3 + h\rho + \beta_3\rho g) + \rho^2(g + \beta_3) + \beta_3^2(g + \rho) + s(\beta_3 + \rho) + hg \\ n_2 &= (g + \beta_3 + \rho) \\ n_4 &= (h\beta_3 + h\rho + \beta_3\rho g) \\ n_6 &= 2\rho h\beta_3(\rho + g + \beta_3) + \rho^2(\beta_3 s + \beta_3^2 g + hg) + \beta_3^2(\rho s + hg) + h^2(\beta_3 + \rho) \end{aligned} \tag{71}$$

and

$$\begin{aligned} &(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)(z_1 + z_4)(z_2 + z_4)(z_3 + z_4) = -d_1 \\ &d_1 = g(h + \beta_3 g + \beta_3^2)(\rho^2 + \rho g + h)(\beta_3 + \rho) > 0 \\ &d_2 = z_1 z_2 z_3 z_4 = h\beta_3\rho > 0 \end{aligned} \tag{72}$$

It is important that d_1 and d_2 are strictly positive. Then, the components of the $\boldsymbol{P}$ tensor take the form,

$$\begin{aligned} P_{11} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}\left(C_{55}C_{66}\frac{n_0}{d_2} + (C_{22}C_{55} + C_{44}C_{66})n_2 + C_{22}C_{44}n_4\right) \\ P_{12} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}(-C_{55}(C_{12} + C_{66})n_2 - C_{44}(C_{12} + C_{66})n_4) \\ P_{22} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}(C_{11}C_{55}n_2 + (C_{55}C_{66} + C_{11}C_{44})n_4 + C_{44}C_{66}n_6) \\ P_{66} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}\left(C_{11}C_{55}\frac{n_0}{d_2} + (C_{11}C_{44} - 2C_{12}C_{55})n_2 + (C_{22}C_{55} - 2C_{12}C_{44})n_4 + C_{22}C_{44}n_6\right) \\ P_{55} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}\left(C_{11}C_{66}\frac{n_0}{d_2} + (C_{11}C_{22} + C_{66}^2 - (C_{12} + C_{66})^2)n_2 + C_{22}C_{66}n_4\right) \\ P_{44} &= \frac{\rho}{d_1 C_{22} C_{44} C_{66}}(C_{11}C_{66}n_2 + (C_{11}C_{22} + C_{66}^2 - (C_{12} + C_{66})^2)n_4 + C_{22}C_{66}n_6) \end{aligned} \tag{73}$$

Expressions (73) are valid for an elliptic cylinder in any orthotropic material including the degenerate case of transversely isotropic material when $g = 2$, $h = 1$, $\beta_3 = 1$. It is worth noting that in order to use (73) one needs to know only the sum and the product of the roots z_1, z_2 but not their individual values. It is also seen that the P_{ij} are real (not complex) numbers.

Consider now the case of a slit crack that occupies the region,

$$-a_1 \leqslant x_1 \leqslant a_1 \quad x_2 = 0 \quad -\infty \leqslant x_3 \leqslant \infty \tag{74}$$

The $\boldsymbol{P}$ tensor may be obtained from (73) by letting $\rho = a_1/a_2 \to \infty$. In this case $\rho n_0/(d_1 d_2) \to 0$, $\rho n_2/d_1 \to 0$, $\rho n_4/d_1 \to 0$, and $\rho n_6/d_1 \to 1$. Thus, we obtain,

$$P_{22} = \frac{1}{C_{22}} \quad P_{44} = \frac{1}{C_{44}} \quad P_{66} = \frac{1}{C_{66}} \tag{75}$$

and other P_{ij} are equal to zero.

Due to cumbersome nature of (73) the derivative of the $\boldsymbol{P}$ tensor with respect to parameter t is at best evaluated numerically as,

$$\dot{P}_{ij} = (P_{ij}(t + \Delta t) - P_{ij}(t))/\Delta t \tag{76}$$

where Δt is the sufficiently small increment of t. In Appendix A we illustrate how one can evaluate $\dot{P}_{ij}$ based on the general formula (59) for the general case of anisotropic media.

4.3. Cylindrical inclusion in a transversely isotropic material

We now apply results obtained in Section 4.1 to transversely isotropic materials. Let x_3 be the axis of elastic symmetry coincident with the axis of the cylindrical inclusion. By using relations (34), it may be easily shown that for a transversely isotropic material the z_v, $v = 1, 2, 3$ are all equal to i. Thus,

$$|\Gamma(z)| = |T|(z - \mathrm{i})^3(z + \mathrm{i})^3 = C_{22}C_{44}C_{66}(z - \mathrm{i})^3(z + \mathrm{i})^3 \tag{77}$$

For a long circular cylinder in a transversely isotropic material analytical expressions for the $\boldsymbol{P}$ tensor has been derived by Walpole (1969). Therefore, it is much simpler to differentiate them with respect to parameter t directly rather than employ (55). (Formula (61) cannot be used since $z_v = \mathrm{i}$.) Instead, we derive here coefficients of the $\dot{\boldsymbol{P}}$ tensor for a cylinder of ellipsoidal cross-section, and outline in Appendix A an analogous derivation for more general anisotropic solids. As an example, we evaluate $\dot{\mathscr{M}}_{1111}$ from (55).

With the use of (77) we have,

$$\dot{\mathscr{M}}_{1111} = -4\int_{-\infty}^{\infty} \frac{\widehat{\Gamma}_{1s}(z)\dot{\Gamma}_{st}(z)\widehat{\Gamma}_{t1}(z)}{(\rho + z^2/\rho)|T|^2(z - \mathrm{i})^6(z + \mathrm{i})^6}\Lambda_{11}(z)\,\mathrm{d}z \tag{78}$$

By expanding (78) in accordance with (62)–(65), we obtain,

$$\begin{aligned}\dot{\mathscr{M}}_{1111} &= -4\int_{-\infty}^{\infty} \frac{\widehat{\Gamma}_{11}(z)\widehat{\Gamma}_{11}(z)\dot{\Gamma}_{11}(z) + \widehat{\Gamma}_{12}(z)\widehat{\Gamma}_{12}(z)\dot{\Gamma}_{22}(z) + 2\widehat{\Gamma}_{11}(z)\widehat{\Gamma}_{12}(z)\dot{\Gamma}_{12}(z)}{(\rho + z^2/\rho)|T|^2(z - \mathrm{i})^6(z + \mathrm{i})^6}\,\mathrm{d}z \\ &= -4\int_{-\infty}^{\infty} \frac{(C_{66} + z^2C_{22})^2(\dot{C}_{11} + z^2\dot{C}_{66}) + z^2(C_{12} + C_{66})^2(\dot{C}_{66} + z^2\dot{C}_{22})}{(\rho + z^2/\rho)C_{22}^2C_{66}^2(z - \mathrm{i})^4(z + \mathrm{i})^4} \\ &\quad - \frac{2z^2(C_{66} + z^2C_{22})(C_{12} + C_{66})(\dot{C}_{12} + \dot{C}_{66})}{(\rho + z^2/\rho)C_{22}^2C_{66}^2(z - \mathrm{i})^4(z + \mathrm{i})^4}\,\mathrm{d}z\end{aligned} \tag{79}$$

since for a transversely isotropic material $C_{55} + z^2C_{44} = C_{44}(1 + z^2)$. Evidently, $z = \mathrm{i}$ is the pole of the integrand in (79) of multiplicity 4, and $z = \rho\mathrm{i}$ is the pole of the multiplicity 1.

The theorem of residues furnishes the value of $\dot{\mathscr{M}}_{1111}$ as,

$$\dot{\mathscr{M}}_{1111} = -\frac{4\pi \mathrm{i}}{3}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\{\ldots(z-\mathrm{i})^4\}|_{z=\mathrm{i}} - 8\pi \mathrm{i}\{\ldots(z-\rho \mathrm{i})\}|_{z=\rho \mathrm{i}} \tag{80}$$

where ... represents the fractional expression under the sign of the integral in (79). After lengthy algebraic manipulations, (79) reduces to,

$$\dot{\mathscr{M}}_{1111} = -2\pi\frac{C_{66}^2\dot{C}_{22}(\rho+2) + C_{22}^2\dot{C}_{66}\rho}{C_{22}^2C_{66}^2(1+\rho)^2} \tag{81}$$

Then, from (19) and with the use of contracted notation P_{st} for components P_{ijkl} we have,

$$\dot{P}_{11} = \dot{P}_{1111} = -\frac{1}{2}\frac{C_{66}^2\dot{C}_{22}(\rho+2) + C_{22}^2\dot{C}_{66}\rho}{C_{22}^2C_{66}^2(1+\rho)^2} \quad \rho = a_1/a_2 \tag{82}$$

Other non-zero components $\dot{P}_{st}$ can be found in a similar manner,

$$\begin{aligned}
\dot{P}_{22} &= \dot{P}_{2222} = -\frac{\rho}{2}\frac{(C_{66}^2\dot{C}_{22}(1+2\rho) + C_{22}^2\dot{C}_{66})}{C_{22}^2C_{66}^2(1+\rho)^2} \\
\dot{P}_{12} &= \dot{P}_{1122} = -\frac{\rho}{2}\frac{(C_{66}^2\dot{C}_{22} - C_{22}^2\dot{C}_{66})}{C_{22}^2C_{66}^2(1+\rho)^2} \\
\dot{P}_{66} &= 4\dot{P}_{1212} = -\frac{2C_{66}^2\dot{C}_{22}\rho + C_{22}^2\dot{C}_{66}(1+\rho^2)}{C_{22}^2C_{66}^2(1+\rho)^2} \\
\dot{P}_{44} &= 4\dot{P}_{2323} = -\frac{\rho\dot{C}_{44}}{C_{44}^2(1+\rho)} \quad \dot{P}_{55} = 4\dot{P}_{1313} = -\frac{\dot{C}_{44}}{C_{44}^2(1+\rho)}
\end{aligned} \tag{83}$$

For completeness, we give expressions for the non-zero components of $\boldsymbol{P}$ tensor,

$$\begin{aligned}
P_{11} &= P_{1111} = \frac{1}{2}\frac{\rho(C_{22}+C_{66}) + 2C_{66}}{C_{22}C_{66}(1+\rho)^2} \quad \rho = a_1/a_2 \\
P_{22} &= P_{2222} = \frac{\rho}{2}\frac{(C_{22}+C_{66}(1+2\rho))}{C_{22}C_{66}(1+\rho)^2} \\
P_{12} &= P_{1122} = \frac{\rho}{2}\frac{C_{66}-C_{22}}{C_{22}C_{66}(1+\rho)^2} \\
P_{66} &= 4P_{1212} = \frac{C_{22}(1+\rho^2) + 2\rho C_{66}}{C_{22}C_{66}(1+\rho)^2} \\
P_{44} &= 4P_{2323} = \frac{1}{C_{44}}\frac{\rho}{1+\rho} \\
P_{55} &= 4P_{1313} = \frac{1}{C_{44}}\frac{1}{1+\rho}
\end{aligned} \tag{84}$$

4.4. Spherical inclusion in an isotropic medium

Consider now the case of spherical inclusions, i.e., $a_1 = a_2 = a_3$ in (1). The tensor $\mathscr{M}_{kijl}$ for points $\boldsymbol{x}$ inside the spherical inclusion follows from (16),

$$\mathscr{M}_{kijl} = \int_{S(\boldsymbol{\omega})} \Gamma_{kj}^{-1}(\boldsymbol{\omega})\omega_i\omega_l \,\mathrm{d}S(\boldsymbol{\omega}) \tag{85}$$

The elastic constants C_{ijkl} for an isotropic material are,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{86}$$

where λ and μ are the Lame constants. For the points on the unit sphere S^2, the tensor $\boldsymbol{\Gamma}(\omega_1, \omega_2, \omega_3)$, defined by (7), becomes,

$$\boldsymbol{\Gamma} = \begin{pmatrix} (\lambda+\mu)\omega_1^2 + \mu & (\lambda+\mu)\omega_1\omega_2 & (\lambda+\mu)\omega_1\omega_3 \\ (\lambda+\mu)\omega_1\omega_2 & (\lambda+\mu)\omega_2^2 + \mu & (\lambda+\mu)\omega_2\omega_3 \\ (\lambda+\mu)\omega_1\omega_3 & (\lambda+\mu)\omega_2\omega_3 & (\lambda+\mu)\omega_3^2 + \mu \end{pmatrix} \tag{87}$$

Its inverse, again for the points on the unit sphere S^2, may be expressed in the form,

$$\boldsymbol{\Gamma}^{-1} = \begin{pmatrix} -\omega_1^2\gamma + 1/\mu & -\omega_1\omega_2\gamma & -\omega_1\omega_3\gamma \\ -\omega_1\omega_2\gamma & -\omega_2^2\gamma + 1/\mu & -\omega_2\omega_3\gamma \\ -\omega_1\omega_3\gamma & -\omega_2\omega_3\gamma & -\omega_3^2\gamma + 1/\mu \end{pmatrix} \tag{88}$$

where

$$\gamma = \frac{\lambda+\mu}{\mu(\lambda+2\mu)}$$

Since the expression (88) of the inverse of $\boldsymbol{\Gamma}$ for isotropic materials is quite simple, the derivative of the $\boldsymbol{P}$ tensor with respect to the parameter t can be determined more easily by direct differentiation of the integrand in (85) rather than by employing (21). With the use of spherical coordinates, we have from (85),

$$\dot{\mathscr{M}}_{kijl} = 2\int_0^{\pi} \sin\theta \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{kj}^{-1}(\omega_1, \omega_2, \omega_3)\omega_i\omega_l \,\mathrm{d}\psi\,\mathrm{d}\theta \tag{89}$$

where $\omega_1 = \sin\theta\cos\psi$, $\omega_2 = \sin\theta\sin\psi$, $\omega_3 = \cos\theta$. The components of the $\dot{\boldsymbol{P}}$ tensor for a spherical inclusion in an isotropic medium are now found from (19) and (89) as,

$$\begin{aligned} \dot{P}_{11} &= -\frac{\mu^2(3\dot{\lambda} + 14\dot{\mu}) + 2\dot{\mu}(\lambda^2 + 4\mu\lambda)}{15\mu^2(\lambda+2\mu)^2} \\ \dot{P}_{12} &= \frac{\mu^2(2\dot{\mu} - \dot{\lambda}) + \dot{\mu}(\lambda^2 + 4\mu\lambda)}{15\mu^2(\lambda+2\mu)^2} \\ \dot{P}_{44} &= -\frac{4\mu^2(\dot{\lambda} + 8\dot{\mu}) + 6\dot{\mu}(\lambda^2 + 4\mu\lambda)}{15\mu^2(\lambda+2\mu)^2} \\ \dot{P}_{11} &= \dot{P}_{22} = \dot{P}_{33}, \quad \dot{P}_{44} = \dot{P}_{55} = \dot{P}_{66}, \quad \dot{P}_{12} = \dot{P}_{13} = \dot{P}_{23} \end{aligned} \tag{90}$$

The components of $\boldsymbol{P}$ tensor are,

$$\begin{aligned} P_{11} &= \frac{7\mu + 2\lambda}{15\mu(\lambda+2\mu)} \\ P_{12} &= \frac{\lambda+\mu}{-15\mu(\lambda+2\mu)} \\ P_{44} &= \frac{2(3\lambda + 8\mu)}{15\mu(\lambda+2\mu)} \\ P_{11} &= P_{22} = P_{33}, \qquad P_{44} = P_{55} = P_{66}, \qquad P_{12} = P_{13} = P_{23} \end{aligned} \tag{91}$$

4.5. Disk in an orthotropic medium

For a thin disk,

$$x_1^2/a_1^2 + x_2^2/a_2^2 \leqslant 1 \quad x_3 = 0 \tag{92}$$

This may be regarded as a limiting case of the ellipsoid (1), by letting $a_3 \to 0$.

It has been shown by Kinoshita and Mura (1971) that as $a_3 \to 0$ the tensor (16) tends to,

$$\mathscr{M}_{kijl} = 4\pi \Gamma_{kj}^{-1}(\boldsymbol{e}_3)\delta_{3i}\delta_{3l} \tag{93}$$

where $\boldsymbol{e}_3 = (0\ 0\ 1)^{\mathrm{T}}$ and Γ_{kj} is defined in (7). Expression (93) remains valid for arbitrary degree of anisotropy of $\boldsymbol{C}$ and for arbitrary a_1/a_2.

Consider a thin disk (92) in an orthotropic medium with x_3 being one of the axes of elastic symmetry. One can find easily that $\Gamma_{kj}^{-1}(\boldsymbol{e}_3)$, required in (93), is given by,

$$\boldsymbol{\Gamma}^{-1}(\boldsymbol{e}_3) = \begin{pmatrix} 1/C_{1313} & 0 & 0 \\ 0 & 1/C_{2323} & 0 \\ 0 & 0 & 1/C_{3333} \end{pmatrix} \tag{94}$$

By differentiating (93) we have,

$$\dot{\mathscr{M}}_{kijl} = 4\pi \frac{\mathrm{d}}{\mathrm{d}t} \Gamma_{kj}^{-1}(\boldsymbol{e}_3)\delta_{3i}\delta_{3l} \tag{95}$$

Now, with the use of contracted notation, the only non-zero components of the $\dot{\boldsymbol{P}}$ tensor are found as,

$$\dot{P}_{33} = -\frac{\dot{C}_{33}}{C_{33}^2} \quad \dot{P}_{44} = -\frac{\dot{C}_{44}}{C_{44}^2} \quad \dot{P}_{55} = -\frac{\dot{C}_{55}}{C_{55}^2} \tag{96}$$

The non-zero components of the $\boldsymbol{P}$ tensor are,

$$P_{33} = \frac{1}{C_{33}} \quad P_{44} = \frac{1}{C_{44}} \quad P_{55} = \frac{1}{C_{55}} \tag{97}$$

5. Applications

5.1. The inhomogeneity problem

Consider now an inhomogeneity of stiffness $\boldsymbol{C}_r$, located in the ellipsoidal domain Ω, defined in Eq. (1), that is contained in a large volume V_0 of a comparison medium of stiffness $\boldsymbol{C}$. This volume is subjected to a certain uniform image strain $\boldsymbol{\epsilon}^{\mathrm{I}}$, applied at the remote boundaries of V_0 (problem (b)). Consider also a related problem (a), for a transformed homogeneous inclusion in the domain Ω, loaded by a uniform eigenstrain $\boldsymbol{\mu}^{\mathrm{a}}$, and embedded in the volume V_0 of stiffness $\boldsymbol{C}$. As in problem (b), a uniform image strain $\boldsymbol{\epsilon}^{\mathrm{I}}$ is applied at the boundary V_0.

Comparing the local strains and stresses in Ω we have,

$$\boldsymbol{\epsilon}_r^{\mathrm{a}} = \boldsymbol{S}\boldsymbol{\mu}^{\mathrm{a}} + \boldsymbol{\epsilon}^{\mathrm{I}} = \boldsymbol{\epsilon}_r^{\mathrm{b}} = \boldsymbol{\mathscr{A}}_r \boldsymbol{\epsilon}^{\mathrm{I}} \tag{98}$$

$$\boldsymbol{\sigma}_r^{\mathrm{a}} = \boldsymbol{C}(\boldsymbol{\epsilon}_r^{\mathrm{a}} - \boldsymbol{\mu}^{\mathrm{a}}) = \boldsymbol{\sigma}_r^{\mathrm{b}} = \boldsymbol{C}_r \boldsymbol{\mathscr{A}}_r \boldsymbol{\epsilon}^{\mathrm{I}} \tag{99}$$

where $\boldsymbol{\mathscr{A}}_r$ is yet unknown partial mechanical strain concentration factor. The first relation yields the equivalent eigenstrain as,

$$\boldsymbol{\mu}_r^{\mathrm{a}} = \boldsymbol{S}^{-1}(\mathscr{A}_r - \boldsymbol{I})\boldsymbol{\epsilon}^{\mathrm{I}} \tag{100}$$

which, when substituted into the Eq. (99), provides the strain concentration factor value as,

$$\mathscr{A}_r^{-1} = (\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{C} - \boldsymbol{C}_r)) = \boldsymbol{P}_r\boldsymbol{P}^{-1} \tag{101}$$

where $\boldsymbol{P} = \boldsymbol{S}\boldsymbol{C}^{-1} = (\boldsymbol{C}^* + \boldsymbol{C})^{-1}$ and $\boldsymbol{P}_r = (\boldsymbol{P}^{-1} - \boldsymbol{C} + \boldsymbol{C}_r)^{-1} = (\boldsymbol{C}^* + \boldsymbol{C}_r)^{-1}$ and $\boldsymbol{C}^*$ is the constraint tensor that represents the stiffness of a uniformly deformed ellipsoidal cavity containing an inhomogeneity $\boldsymbol{C}_r$. Substituting (101) into (100) we have,

$$\boldsymbol{\mu}_r^{\mathrm{a}} = \boldsymbol{C}^{-1}\boldsymbol{P}^{-1}(\boldsymbol{P}_r - \boldsymbol{P})\boldsymbol{P}^{-1}\boldsymbol{\epsilon}^{\mathrm{I}} \tag{102}$$

Rate forms of (98) and (99) are,

$$\dot{\boldsymbol{\epsilon}}_r^{\mathrm{a}} = \dot{\boldsymbol{S}}\boldsymbol{\mu}^{\mathrm{a}} + \boldsymbol{S}\dot{\boldsymbol{\mu}}^{\mathrm{a}} + \dot{\boldsymbol{\epsilon}}^{\mathrm{I}} = \dot{\boldsymbol{\epsilon}}_r^{\mathrm{b}} = \dot{\mathscr{A}}_r\boldsymbol{\epsilon}^{\mathrm{I}} + \mathscr{A}_r\dot{\boldsymbol{\epsilon}}^{\mathrm{I}} \tag{103}$$

$$\dot{\boldsymbol{\sigma}}_r^{\mathrm{a}} = \dot{\boldsymbol{C}}(\boldsymbol{\epsilon}_r^{\mathrm{a}} - \boldsymbol{\mu}^{\mathrm{a}}) + \boldsymbol{C}(\dot{\boldsymbol{\epsilon}}_r^{\mathrm{a}} - \dot{\boldsymbol{\mu}}^{\mathrm{a}}) = \dot{\boldsymbol{\sigma}}_r^{\mathrm{b}} = (\dot{\boldsymbol{C}}_r\mathscr{A}_r + \boldsymbol{C}_r\dot{\mathscr{A}}_r)\boldsymbol{\epsilon}^{\mathrm{I}} + \boldsymbol{C}_r\mathscr{A}_r\dot{\boldsymbol{\epsilon}}^{\mathrm{I}} \tag{104}$$

Since the eigenstrain (100) and the concentration factor (101) satisfy the Eqs. (98) and (99), their derivatives $\dot{\boldsymbol{\mu}}_r^{\mathrm{a}}$ and $\dot{\mathscr{A}}_r$ can be shown to satisfy the rate form of these equations, i.e., (103) and (104).

In summary, the rates of the strain and stress fields in the inhomogeneity $\boldsymbol{C}_r$ residing in comparison medium $\boldsymbol{C}$ that is loaded remotely by $\boldsymbol{\epsilon}^{\mathrm{I}}$ are,

$$\dot{\boldsymbol{\epsilon}}_r = \dot{\mathscr{A}}_r\boldsymbol{\epsilon}^{\mathrm{I}} + \mathscr{A}_r\dot{\boldsymbol{\epsilon}}^{\mathrm{I}} \tag{105}$$

$$\dot{\boldsymbol{\sigma}}_r = (\dot{\boldsymbol{C}}_r\mathscr{A}_r + \boldsymbol{C}_r\dot{\mathscr{A}}_r)\boldsymbol{\epsilon}^{\mathrm{I}} + \boldsymbol{C}_r\mathscr{A}_r\dot{\boldsymbol{\epsilon}}^{\mathrm{I}} \tag{106}$$

The derivative of the $\mathscr{A}_r$ is found from (101) as,

$$\dot{\mathscr{A}}_r = \mathscr{A}_r[\dot{\boldsymbol{P}}(\boldsymbol{C} - \boldsymbol{C}_r) + \boldsymbol{P}(\dot{\boldsymbol{C}} - \dot{\boldsymbol{C}}_r)]\mathscr{A}_r \tag{107}$$

5.2. Estimates of overall stresses and stress rates

The overall stiffness of a composite aggregate consisting of phases $r = 1, 2, \ldots, n$ can be found using the standard form (Hill, 1965; Walpole, 1966) $\boldsymbol{L} = \sum_{r=1}^{n} c_r\boldsymbol{C}_r\boldsymbol{A}_r$, where $\boldsymbol{A}_r$ is the total concentration factor $\boldsymbol{A}_r = \mathscr{A}_r(\sum_{s=1}^{n}\mathscr{A}_s)^{-1}$, and c_r is the volume fraction of the phase r. The overall stress and stress rate supported by a composite with evolving moduli, under applied strain $\boldsymbol{\epsilon}^0$ and strain rate $\dot{\boldsymbol{\epsilon}}^0$ are,

$$\begin{aligned} \boldsymbol{\sigma} &= \sum_{r=1}^{n} c_r\boldsymbol{C}_r\boldsymbol{A}_r\boldsymbol{\epsilon}^0 \\ \dot{\boldsymbol{\sigma}} &= \sum_{r=1}^{n} c_r(\dot{\boldsymbol{C}}_r\boldsymbol{A}_r + \boldsymbol{C}_r\dot{\boldsymbol{A}}_r)\boldsymbol{\epsilon}^0 + c_r\boldsymbol{C}_r\boldsymbol{A}_r\dot{\boldsymbol{\epsilon}}^0 \end{aligned} \tag{108}$$

The stiffness $\boldsymbol{C}$ of the comparison medium is chosen according to the selected averaging method. For example, $\boldsymbol{C} = \boldsymbol{C}_1$ provides the Mori-Tanaka estimate for a composite with matrix $\boldsymbol{C}_1$. The choice $\boldsymbol{C} = \boldsymbol{L}$ implies the self-consistent estimate which would be rather difficult to implement in the present context. Other admissible choices of $\boldsymbol{C}$ in terms of known phase properties can be found in Dvorak and Srinivas (1999).

6. Conclusions

The results present derivations of solutions of transformed homogeneous inclusion problems in anisotropic solids that have variable elastic moduli $\boldsymbol{C}(t)$, dependent on an evolution parameter t, and changing at a prescribed rate $\partial\boldsymbol{C}(t)/\partial t$. The results are found in terms of rates $\partial\boldsymbol{P}(t)/\partial t$ of the $\boldsymbol{P}$ tensors for several useful shapes of ellipsoidal inclusions in anisotropic solids with symmetries of typical composite material systems, and are derived in explicit or closed form. The well-known connections with the Eshelby tensor $\boldsymbol{S}(t) = \boldsymbol{P}(t)\boldsymbol{C}(t)$ then provides the rate forms $\partial\boldsymbol{S}(t)/\partial t = (\partial\boldsymbol{P}(t)/\partial t)\boldsymbol{C}(t) + \boldsymbol{P}(t)(\partial\boldsymbol{C}(t)/\partial t)$, which are utilized in solving inhomogeneity problems in heterogeneous solids with varying constituent stiffness coefficients. Extension of these results to evaluation of overall moduli rates by standard averaging methods is also outlined.

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Appendix A

Here we elucidate how one can evaluate the tensors (59) and (61) for a cylinder in an arbitrary anisotropic medium. In this case components of the matrix $\boldsymbol{\Gamma}(z)$ with the choice of the unit vectors $\hat{\boldsymbol{n}} = \{1 \quad 0 \quad 0\}^{\mathrm{T}}$, $\hat{\boldsymbol{m}} = \{0 \quad 1 \quad 0\}^{\mathrm{T}}$ are given by,

$$\begin{aligned}
&\Gamma_{11} = C_{11} + 2zC_{16} + z^2C_{66} \quad \Gamma_{12} = C_{16} + z(C_{12} + C_{66}) + z^2C_{26} \\
&\Gamma_{13} = C_{15} + z(C_{14} + C_{56}) + z^2C_{46} \quad \Gamma_{22} = C_{66} + 2zC_{26} + z^2C_{22} \\
&\Gamma_{23} = C_{56} + z(C_{46} + C_{25}) + z^2C_{24} \quad \Gamma_{33} = C_{55} + 2zC_{45} + z^2C_{44}
\end{aligned} \tag{A.1}$$

The derivative of the matrix $\boldsymbol{\Gamma}$ with respect to parameter t (and the matrix itself) is a polynomial in z of the second degree and thus can be represented as,

$$\dot{\boldsymbol{\Gamma}}(z) = \sum_{l=0}^{2} z^l \dot{\boldsymbol{\Gamma}}^l \tag{A.2}$$

where $\dot{\boldsymbol{\Gamma}}^l$ are the real symmetric matrices dependent only on the derivatives $\dot{C}_{ijkl}$ but not on z. Similarly, the adjoint matrix $\hat{\boldsymbol{\Gamma}}(z)$ can be represented as a polynomial in z of degree four,

$$\hat{\boldsymbol{\Gamma}}(z) = \sum_{n=0}^{4} z^n \hat{\boldsymbol{\Gamma}}^n \tag{A.3}$$

where $\hat{\boldsymbol{\Gamma}}^n$ are the real symmetric matrices independent of z. We list some of the components of the matrices $\hat{\boldsymbol{\Gamma}}^n$ below,

$$\begin{aligned}
&\widehat{\Gamma}^0_{11} = C_{66}C_{55} - C_{56}^2 \quad \widehat{\Gamma}^0_{12} = -C_{16}C_{55} + C_{15}C_{56} \\
&\widehat{\Gamma}^0_{13} = C_{16}C_{56} - C_{15}C_{66} \quad \widehat{\Gamma}^0_{22} = C_{11}C_{55} - C_{15}^2 \\
&\widehat{\Gamma}^0_{23} = C_{15}C_{16} - C_{11}C_{56} \quad \widehat{\Gamma}^0_{33} = C_{11}C_{66} - C_{16}^2
\end{aligned} \tag{A.4}$$

$$\begin{aligned}\widehat{\Gamma}^1_{11} &= 2C_{66}C_{45} + 2C_{26}C_{55} - 2C_{56}(C_{46} + C_{25})\\ \widehat{\Gamma}^1_{12} &= -C_{55}(C_{66} + C_{12}) + C_{56}(C_{56} + C_{14}) + C_{15}(C_{46} + C_{25}) - 2C_{16}C_{45}\end{aligned} \tag{A.5}$$

$$\begin{aligned}\widehat{\Gamma}^2_{11} &= C_{66}C_{44} + 4C_{26}C_{45} + C_{22}C_{55} - 2C_{56}C_{24} - C^2_{46} - 2C_{46}C_{25} - C^2_{25}\\ \widehat{\Gamma}^2_{12} &= C_{56}(2C_{46} + C_{25}) + C_{14}(C_{46} + C_{25}) - 2C_{45}(C_{12} + C_{66}) - C_{26}C_{55} + C_{15}C_{24} - C_{16}C_{44}\end{aligned} \tag{A.6}$$

$$\begin{aligned}\widehat{\Gamma}^3_{11} &= 2C_{26}C_{44} + 2C_{22}C_{45} - 2C_{46}C_{24} - 2C_{25}C_{24}\\ \widehat{\Gamma}^3_{12} &= C_{46}(C_{46} + C_{25}) - 2C_{26}C_{45} - C_{44}(C_{66} + C_{12}) + C_{56}C_{24} + C_{14}C_{24}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\widehat{\Gamma}^4_{11} &= C_{22}C_{44} - C^2_{24} \quad \widehat{\Gamma}^4_{12} = C_{46}C_{24} - C_{26}C_{44}\\ \widehat{\Gamma}^4_{13} &= C_{26}C_{24} - C_{46}C_{22} \quad \widehat{\Gamma}^4_{22} = C_{66}C_{44} - C^2_{46}\\ \widehat{\Gamma}^4_{23} &= C_{26}C_{46} - C_{66}C_{24} \quad \widehat{\Gamma}^4_{33} = C_{66}C_{22} - C^2_{26}\end{aligned} \tag{A.8}$$

Using definitions (A.2) and (A.3) in (59) we arrive at the following formula, suitable for numerical determination of $\dot{\mathscr{M}}$,

$$\begin{aligned}\dot{\mathscr{M}}_{k\alpha j\beta} = &-\frac{8\pi\mathrm{i}}{|T|^2}\sum_{s=1}^{3}\sum_{t=1}^{3}\sum_{l=0}^{2}\sum_{m=0}^{4}\sum_{n=0}^{4}\left\{\sum_{v=1}^{3}\left[\widehat{\Gamma}^n_{ks}\widehat{\Gamma}^m_{tj}\dot{\Gamma}^l_{st}\delta z_v^{\delta-1}\left(\rho+\frac{z_v^2}{\rho}\right)t_v^2(z_v)\right.\right.\\ &\left.\left.-\widehat{\Gamma}^n_{ks}\widehat{\Gamma}^m_{tj}\dot{\Gamma}^l_{st}z_v^{\delta}\left(2\frac{z_v}{\rho}t_v^2(z_v)+2\left(\rho+\frac{z_v^2}{\rho}\right)t_v(z_v)t_v'(z_v)\right)\right]\Big/\left[\left(\rho+\frac{z_v^2}{\rho}\right)^2 t_v^4(z_v)\right]\right\}\\ &+\{\widehat{\Gamma}^n_{ks}\widehat{\Gamma}^m_{tj}\dot{\Gamma}^l_{st}(\rho\mathrm{i})^{\delta}/[2\mathrm{i}f^2(\rho\mathrm{i})]\},\\ &\alpha,\beta=1,2,\ \ k,j=1,2,3,\ \ \delta=n+m+l+\alpha+\beta-2,\ \ \rho=a_1/a_2,\ \ z_v\neq \mathrm{i},\rho\mathrm{i},\\ &\dot{\mathscr{M}}_{k\alpha j\beta}=0 \quad \text{if } \alpha=3 \text{ or } \beta=3\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}f(z) &= (z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)(z-z_3)(z-\bar{z}_3)\\ t_1(z) &= (z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)(z-z_3)(z-\bar{z}_3)\\ t_1'(z) &= t_1(z)\left\{\frac{1}{z-\bar{z}_1}+\frac{1}{z-z_2}+\frac{1}{z-\bar{z}_2}+\frac{1}{z-z_3}+\frac{1}{z-\bar{z}_3}\right\}\end{aligned} \tag{A.10}$$

and all other t_v are obtained from (A.10) by a cyclic permutation of the subscripts, and the determinant of the matrix $\boldsymbol{T}$ is given by,

$$|T| = C_{22}C_{44}C_{66} - C_{66}C^2_{24} - C^2_{26}C_{44} + 2C_{26}C_{46}C_{24} - C^2_{46}C_{22} \tag{A.11}$$

The Stroh eigenvalues z_v are found numerically by solving (56). The tensor $\dot{\boldsymbol{P}}$ is found from (19). Note that the expressions (A.9) become singular when the elastic symmetry of the medium tends to transversely isotropic. Loss of accuracy may occur because of the division by very small quantities such as $t_v(z_v)$ for any ρ and $f(\rho\mathrm{i})$ for $\rho = 1$. In this case the expressions (82) and (83) must be used.

For completeness, we give expressions for the components of the $\mathscr{M}$ tensor, as it follows from (61),

$$\mathscr{M}_{k\alpha j\beta} = \frac{8\pi i}{|T|}\sum_{n=0}^{4}\left\{\sum_{v=1}^{3}\widehat{\Gamma}^{n}_{kj}z_v^{\delta}\Big/\left[\left(\rho+\frac{z_v^2}{\rho}\right)t_v(z_v)\right]\right\} + \{\widehat{\Gamma}^{n}_{kj}(\rho i)^{\delta}/[2if(\rho i)]\}$$

$$\alpha,\beta = 1,2, \quad k,j = 1,2,3 \tag{A.12}$$

$$\delta = n+\alpha+\beta-2, \quad \rho = a_1/a_2, \quad z_v \neq i, \rho i$$

$$\mathscr{M}_{k\alpha j\beta} = 0 \quad \text{if } \alpha = 3 \text{ or } \beta = 3$$

The $\boldsymbol{P}$ tensor is found from (18).

Appendix B

Components of the Eshelby tensor $\boldsymbol{S}$, defined by (3), and their derivatives for certain inclusion shapes are listed below. Derivatives of the Eshelby tensor are found from the connection $\boldsymbol{S} = \boldsymbol{PC}$, i.e.,

$$\dot{S}_{ijkl} = \dot{P}_{ijmn}C_{mnkl} + P_{ijmn}\dot{C}_{mnkl} \tag{B.1}$$

Elliptic cylinders (39) in a transversely isotropic medium (x_3 is the axis of rotational symmetry):

$$\begin{aligned}
&S_{1111} = \frac{C_{22}(1+2\rho)-\rho C_{66}}{C_{22}(1+\rho)^2} \quad S_{2222} = \frac{\rho(C_{22}(2+\rho)-C_{66})}{C_{22}(1+\rho)^2} \quad \rho = a_1/a_2\\
&S_{1122} = \frac{C_{22}-C_{66}(2+\rho)}{C_{22}(1+\rho)^2} \quad S_{2211} = \frac{\rho(\rho C_{22}-C_{66}(1+2\rho))}{C_{22}(1+\rho)^2}\\
&S_{1133} = \frac{C_{13}}{(1+\rho)C_{22}} \quad S_{2233} = \frac{\rho C_{13}}{(1+\rho)C_{22}}\\
&S_{3333} = 0 \quad S_{3311} = 0 \quad S_{3322} = 0\\
&S_{1212} = \frac{1}{2}\frac{C_{22}(1+\rho^2)+2C_{66}\rho}{C_{22}(1+\rho)^2}\\
&S_{1313} = \frac{1}{2(1+\rho)} \quad S_{2323} = \frac{\rho}{2(1+\rho)}
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\dot{S}_{1111} = \dot{S}_{2222} = \frac{\rho(C_{66}\dot{C}_{22}-\dot{C}_{66}C_{22})}{C_{22}^2(1+\rho)^2}\\
&\dot{S}_{1122} = \frac{(2+\rho)(C_{66}\dot{C}_{22}-\dot{C}_{66}C_{22})}{C_{22}^2(1+\rho)^2} \quad \dot{S}_{2211} = \frac{\rho(1+2\rho)(C_{66}\dot{C}_{22}-\dot{C}_{66}C_{22})}{C_{22}^2(1+\rho)^2}\\
&\dot{S}_{1133} = \frac{C_{22}\dot{C}_{13}-C_{13}\dot{C}_{22}}{C_{22}^2(1+\rho)} \quad \dot{S}_{2233} = \frac{\rho(C_{22}\dot{C}_{13}-C_{13}\dot{C}_{22})}{C_{22}^2(1+\rho)}\\
&\dot{S}_{1212} = \frac{\rho(\dot{C}_{66}C_{22}-C_{66}\dot{C}_{22})}{C_{22}^2(1+\rho)^2}
\end{aligned} \tag{B.3}$$

Spherical inclusions in an isotropic medium with Lame constants λ and μ:

$$\begin{aligned}
&S_{1111} = S_{2222} = S_{3333} = \frac{14\mu + 9\lambda}{15(\lambda + 2\mu)} \quad \dot{S}_{1111} = \frac{4}{15}\frac{\mu\dot{\lambda} - \lambda\dot{\mu}}{(\lambda + 2\mu)^2} \\
&S_{1122} = S_{1133} = S_{2233} = \frac{3\lambda - 2\mu}{15(\lambda + 2\mu)} \quad \dot{S}_{1122} = \frac{8}{15}\frac{\mu\dot{\lambda} - \lambda\dot{\mu}}{(\lambda + 2\mu)^2} \\
&S_{1122} = S_{2211} = S_{3311} = S_{3322} \\
&S_{1212} = S_{2323} = S_{1313} = \frac{3\lambda + 8\mu}{15(\lambda + 2\mu)} \quad \dot{S}_{1212} = \frac{2}{15}\frac{\lambda\dot{\mu} - \mu\dot{\lambda}}{(\lambda + 2\mu)^2}
\end{aligned} \tag{B.4}$$

Disks (92) in an orthotropic medium:

$$\begin{aligned}
&S_{1111} = S_{2222} = S_{1133} = S_{2233} = S_{1122} = S_{2211} = S_{1212} = 0 \\
&S_{3311} = \frac{C_{13}}{C_{33}} \quad \dot{S}_{3311} = \frac{C_{33}\dot{C}_{13} - C_{13}\dot{C}_{33}}{C_{33}^2} \\
&S_{3322} = \frac{C_{23}}{C_{33}} \quad \dot{S}_{3322} = \frac{C_{33}\dot{C}_{23} - C_{23}\dot{C}_{33}}{C_{33}^2} \\
&S_{3333} = 1 \quad S_{2323} = S_{1313} = \tfrac{1}{2} \quad \dot{S}_{3333} = \dot{S}_{2323} = 0
\end{aligned} \tag{B.5}$$

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